

# LONG TIME EXISTENCE OF THE ( $n - 1$ )-PLURISUBHARMONIC FLOW

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ABSTRACT. We consider the  $(n-1)$ -plurisubharmonic flow, suggested by Tosatti-Weinkove, and prove a formula for its maximal time of existence. This includes estimates that will be useful in further investigating the flow.

## 1. INTRODUCTION

Let  $M$  be a compact complex manifold of dimension  $n > 2$  with  $g$  and  $g_0$  Hermitian metrics on  $M$ . We define the associated real  $(1, 1)$ -form

$$\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}$$

which will also refer to as a metric. The  $(n - 1)$ -plurisubharmonic flow is the equation

$$(1.1) \quad \frac{\partial}{\partial t}\omega_t^{n-1} = -(n-1)\text{Ric}^C(\omega_t) \wedge \omega_t^{n-2}, \quad \omega_t|_{t=0} = \omega_0.$$

where  $\text{Ric}^C(\omega_t) = -\sqrt{-1}\partial\bar{\partial}\log\omega_t^n$  is the Chern-Ricci form of  $\omega_t$ . In the case of  $n = 2$ , (1.1) becomes the Chern-Ricci flow (see [8, 9, 10, 13, 17, 20, 23, 24, 27]). This flow was originally suggested by Tosatti-Weinkove in their work on the elliptic Monge-Ampère equation for  $(n - 1)$ -plurisubharmonic forms [25, 26].

We say that a metric  $\omega_0$  is *balanced* [16] if

$$d\omega_0^{n-1} = 0$$

*Gauduchon* [5] if

$$\partial\bar{\partial}\omega_0^{n-1} = 0$$

and *strongly Gauduchon* (recently introduced by Popovici in [18]) if

$$\bar{\partial}\omega_0^{n-1} \text{ is } \partial\text{-exact.}$$

When  $\omega$  is a Kähler metric

$$d\omega = 0$$

then the  $(n - 1)$ -plurisubharmonic flow preserves all three of the above conditions imposed on  $\omega_0$ . If instead  $\omega$  is an *Asthenic-Kähler* metric (see [12])

$$\partial\bar{\partial}\omega^{n-2} = 0$$

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the flow preserves the Gauduchon and strongly Gauduchon conditions, but not necessarily the balanced condition. Indeed, the flow is equivalent to

$$\frac{\partial}{\partial t} \omega_t^{n-1} = -(n-1) \text{Ric}^C(\omega) \wedge \omega^{n-2} + \sqrt{-1} \partial \bar{\partial} \theta(t) \wedge \omega^{n-2}$$

where

$$\theta(t) = \log \frac{\det(g_t)^{n-1}}{\det g^{n-1}}.$$

Defining

$$\Phi_t = \omega_0^{n-1} - t(n-1) \text{Ric}^C(\omega) \wedge \omega^{n-2}$$

we see that a solution to (1.1) is of the form

$$\omega_t^{n-1} = \Phi_t + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2}$$

for some real valued function  $u$  on  $M$ . One can check that if  $\omega$  is Kähler and  $\omega_0$  is balanced (respectively Gauduchon, strongly Gauduchon), then the family of metrics  $\omega_t$  is balanced (respectively Gauduchon, strongly Gauduchon) for all  $t$  along the flow. Similarly for  $\omega$  Astheno-Kähler and  $\omega_0$  Gauduchon or strongly Gauduchon.

We prove the following formula for the maximal time of existence of the flow assuming  $\omega_0$  and  $\omega$  are Hermitian metrics.

**Theorem 1.1.** *Let  $M$  be a compact complex manifold of dimension  $n \geq 3$  and let  $\omega_0$  and  $\omega$  be Hermitian metrics on  $M$ . Then there exists a unique solution of the  $(n-1)$ -plurisubharmonic flow (1.1) on the maximal time interval  $[0, T)$  where*

$$T = \sup \{ t > 0 \mid \exists \psi \in C^\infty(M) \text{ such that } \Phi_t + \sqrt{-1} \partial \bar{\partial} \psi \wedge \omega^{n-2} > 0 \}.$$

Note that if we define an equivalence relation of real  $(n-1, n-1)$ -forms by

$$\Psi \sim \Psi' \iff \Psi = \Psi' + \sqrt{-1} \partial \bar{\partial} \psi \wedge \omega^{n-2} \text{ for some } \psi \in C^\infty(M)$$

then  $T$  depends only on  $\omega$  and the equivalence class of  $\omega_0^{n-1}$ . This is analogous to the result of Tian-Zhang for the Kähler-Ricci flow [22] and of Tosatti-Weinkove for the Chern-Ricci flow [23]. Much like these related results, this theorem suggests that the  $(n-1)$ -plurisubharmonic flow is a natural object of study that reflects the geometry of the manifolds.

Every Hermitian metric is conformal to a Gauduchon metric [5] on a compact complex manifold. However if  $\omega$  is only assumed to be Gauduchon then the  $(n-1)$ -plurisubharmonic flow (1.1) does not preserve the Gauduchon condition of  $\omega_0$ . To alleviate this problem we consider the new flow

$$(1.2) \quad \frac{\partial}{\partial t} \omega_t^{n-1} = -(n-1) \text{Ric}^C(\omega_t) \wedge \omega^{n-2} - (n-1) \text{Re} \left( \sqrt{-1} \partial (\log \omega_t^n) \wedge \bar{\partial} (\omega^{n-2}) \right).$$

If the fixed metric  $\omega$  is Gauduchon and the initial metric  $\omega_0$  is Gauduchon or strongly Gauduchon, so is the solution to (1.2) for as long as it exists. To

see this, we compute as above. A solution to this new flow (1.2) is of the form

$$\omega_t^{n-1} = \hat{\Phi}_t + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2} + \operatorname{Re}(\sqrt{-1}\partial u \wedge \bar{\partial}(\omega^{n-2}))$$

where

$$\hat{\Phi}_t = \omega_0^{n-1} - t(n-1)(\operatorname{Ric}^C(\omega) \wedge \omega^{n-2} + \operatorname{Re}(\sqrt{-1}\partial(\log \omega^n) \wedge \bar{\partial}(\omega^{n-2}))).$$

We conjecture that this flow has a similar theorem for its maximal existence time, but we are currently unable to prove the estimates that would give this result.

**Conjecture 1.2.** *Let  $M$  be a compact complex manifold of dimension  $n \geq 3$ ,  $\omega$  a Gauduchon metric, and  $\omega_0$  a Hermitian metric on  $M$ . Then there exists a unique solution of (1.2) on the maximal time interval  $[0, T)$  where*

$$T = \sup \left\{ t > 0 \mid \begin{array}{l} \exists \psi \in C^\infty(M) \text{ such that} \\ \hat{\Phi}_t + \sqrt{-1}\partial\bar{\partial}\psi \wedge \omega^{n-2} \\ + \operatorname{Re}(\sqrt{-1}\partial\psi \wedge \bar{\partial}(\omega^{n-2})) > 0 \end{array} \right\}.$$

The estimates required to prove the above conjecture are the same as those needed to prove Gauduchon's conjecture:

**Conjecture 1.3.** *(Gauduchon, 1977 [6]) Let  $M$  be a compact complex manifold and let  $\psi$  be a closed real  $(1,1)$ -form on  $M$  with  $[\psi] = c_1^{BC}(M)$ . Then there exists a Gauduchon metric  $\tilde{\omega}$  on  $M$  with*

$$\operatorname{Ric}^C(\tilde{\omega}) = \psi.$$

This is a generalization of the famous Calabi-Yau theorem in Kähler geometry [28]. Popovici [19] and Tosatti-Weinkove [26] have both recently shown that proving Gauduchon's conjecture is equivalent to solving

$$(1.3) \quad \det(\Phi_u) = e^{F+b} \det(\omega^{n-1})$$

with

$$\Phi_u = \omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2} + \operatorname{Re}(\sqrt{-1}\partial u \wedge \bar{\partial}(\omega^{n-2})) > 0$$

with  $\sup_M u = 0$  and  $\omega$  Gauduchon. The missing ingredient for the solution is a second order estimate for  $u$  solving (1.3). Consider (1.3) where we remove the last term in the definition of  $\Phi_u$ :

$$(1.4) \quad \det(\omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2}) = e^{F+b} \det(\omega^{n-1})$$

with

$$\omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2} > 0, \quad \sup_M u = 0$$

Fu-Wang-Wu [3] proved that (1.4) has a smooth solution when  $\omega$  is Kähler and has nonnegative orthogonal bisectional curvature and Tosatti-Weinkove have proven this result with no assumptions on  $\omega$  other than being a Hermitian metric [25, 26]. The estimates of [26] are crucial in the proof of the main theorem which we now summarize.

The general strategy is similar to that of the analogous results for the Kähler-Ricci flow [22] (see also [21]) and Chern-Ricci flow [23]. Note that the flow (1.1) cannot exist beyond  $T$  as defined in the main theorem, so we assume that the flow has a maximal time of existence  $S < T$ . The  $(n-1)$ -plurisubharmonic flow is reduced to the parabolic scalar flow

$$(1.5) \quad \frac{\partial}{\partial t} u = \log \frac{\left( \hat{\omega}_t + \frac{1}{n-1} ((\Delta u)\omega - \sqrt{-1}\partial\bar{\partial}u) \right)^n}{\Omega}, \quad u|_{t=0} = 0$$

with

$$\hat{\omega}_t + \frac{1}{n-1} ((\Delta u)\omega - \sqrt{-1}\partial\bar{\partial}u) > 0$$

on  $[0, S)$ . The maximum principle gives uniform bounds for  $u$ ,  $\dot{u}$ , and the volume form  $\tilde{\omega}_t^n$  where

$$\tilde{\omega}_t = \hat{\omega}_t + \frac{1}{n-1} ((\Delta u)\omega - \sqrt{-1}\partial\bar{\partial}u).$$

We then apply the maximum principle to obtain the estimate

$$\mathrm{tr}_\omega \tilde{\omega}_t \leq C \left( \sup_{M \times [0, S)} |\nabla u|_g^2 + 1 \right)$$

which is the parabolic version of the estimate from [26] and the proof uses many similar elements. Following [25], we use a Liouville theorem and blow-up argument to uniformly bound  $|\nabla u|_g^2$ . Applying the Evans-Kyrlov method (see [7, 15] and [8] in the complex setting for parabolic equations) gives the  $C^{2+\alpha}(M, g)$  estimate and then from standard parabolic theory we produce higher order estimates. This allows us to extend the flow beyond the time  $S$  contradicting the maximality of  $S$ .

## 2. REDUCTION TO MONGE-AMPÈRE AND NOTATION

We define the Christoffel symbols of the Hermitian metric  $g$  in local holomorphic coordinates  $(z^1, \dots, z^n)$  by

$$\Gamma_{ij}^k = g^{k\bar{l}} \partial_i g_{j\bar{l}}$$

and the covariant derivative with respect to  $g$  by

$$\nabla_i a_l = \partial_i a_l - \Gamma_{il}^p a_p.$$

The torsion of  $g$  is the tensor

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k.$$

Note that if  $g$  is a Kähler metric, then  $T_{ij}^k = 0$ . The Chern curvature of  $g$  is

$$R_{k\bar{l}i}{}^p = -\partial_{\bar{l}} \Gamma_{ki}^p$$

and it obeys the usual commutation identities for curvature. For example,

$$[\nabla_i, \nabla_{\bar{j}}] a_l = -R_{i\bar{j}l}{}^p a_p, \quad [\nabla_i, \nabla_{\bar{j}}] a_{\bar{m}} = R_{i\bar{j}}{}^{\bar{q}}{}_{\bar{m}} \bar{a}_{\bar{q}}.$$

We will make use of the commutation formulas

$$(2.6) \quad \begin{aligned} u_{i\bar{j}l} &= u_{il\bar{j}} - u_p R_{l\bar{j}i}^p, \quad u_{p\bar{j}\bar{m}} = u_{p\bar{m}\bar{j}} - \overline{T_{mj}^q} u_{p\bar{q}}, \quad u_{i\bar{q}l} - T_{li}^p u_{p\bar{q}} \\ u_{i\bar{j}\bar{l}m} &= u_{l\bar{m}i\bar{j}} + u_{p\bar{j}} R_{l\bar{m}i}^p - u_{p\bar{m}} R_{i\bar{j}l}^p - T_{li}^p u_{p\bar{m}\bar{j}} - \overline{T_{mj}^q} u_{l\bar{q}i} - T_{il}^p \overline{T_{mj}^q} u_{p\bar{q}}. \end{aligned}$$

The Chern-Ricci form  $\text{Ric}^C(\omega)$  is given by

$$\text{Ric}^C(\omega) = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

where

$$R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\partial_i \partial_{\bar{j}} \log \det g.$$

A real  $(n-1, n-1)$ -form  $\Psi$  is defined to be positive definite if for every nonzero  $(1, 0)$ -form  $\gamma$ ,

$$\Psi \wedge \sqrt{-1} \gamma \wedge \bar{\gamma} \geq 0$$

with equality if and only if  $\gamma = 0$ . The determinant of a  $\Psi$  is given by the determinant of the matrix  $(\Psi_{i\bar{j}})$  where

$$\begin{aligned} \Psi &= (\sqrt{-1})^{n-1} (n-1)! \sum_{i,j} (\text{sgn}(i, j)) \Psi_{i\bar{j}} dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge \hat{dz}^i \wedge dz^i \wedge \dots \\ &\quad \wedge dz^j \wedge d\bar{z}^j \wedge \dots \wedge dz^n \wedge d\bar{z}^n. \end{aligned}$$

Using this formula,

$$\det(\omega^{n-1}) = (\det g)^{n-1}.$$

We say that a constant  $C > 0$  is uniform if it only depends on the initial data for the  $(n-1)$ -plurisubharmonic flow. In our calculations a uniform constant  $C$  may change from line to line.

Now we set up the proof of the main theorem. Suppose that  $S$  is such that  $0 < S < T$ . Then there exists a smooth function  $\psi$  such that

$$(2.7) \quad \Psi_S := \Phi_S + \sqrt{-1} \partial \bar{\partial} \psi \wedge \omega^{n-2} > 0.$$

We define  $\Psi_t$  to be the straight line path from  $\omega_0^{n-1}$  to  $\Psi_S$  on  $[0, S]$

$$(2.8) \quad \begin{aligned} \Psi_t &= \frac{1}{S} ((S-t)\omega_0^{n-1} + t(\Phi_S + \sqrt{-1} \partial \bar{\partial} \psi \wedge \omega^{n-2})) \\ &= \omega_0^{n-1} + t\chi \wedge \omega^{n-2} \end{aligned}$$

where  $\chi = \frac{1}{S} \sqrt{-1} \partial \bar{\partial} \psi - (n-1) \text{Ric}^C(\omega)$ . From its definition, note that  $\Psi_t$  is uniformly bounded in the sense that there exists a uniform constant  $C$  such that

$$(2.9) \quad \frac{1}{C} \omega^{n-1} \leq \Psi_t \leq C \omega^{n-1}$$

on  $M \times [0, S]$ . Define a family of Hermitian metrics  $\hat{\omega}_t$  by

$$\hat{\omega}_t = \frac{1}{(n-1)!} * \Psi_t = \hat{\omega}_0 + \frac{t}{n-1} ((\text{tr}_\omega \chi) \omega - \chi)$$

where  $*$  is the Hodge star operator with respect to  $g$  and

$$\hat{\omega}_0 = \frac{1}{(n-1)!} * \omega_0^{n-1}.$$

From (2.9) we also have

$$(2.10) \quad \frac{1}{C} \omega \leq \hat{\omega}_t \leq C \omega.$$

on  $M \times [0, S]$  for some uniform  $C$ .

Suppose that  $u$  satisfies (1.5)

$$\frac{\partial}{\partial t} u = \log \frac{\left( \hat{\omega}_t + \frac{1}{n-1} ((\Delta u) \omega - \sqrt{-1} \partial \bar{\partial} u) \right)^n}{\Omega}, \quad u|_{t=0} = 0$$

with  $\hat{\omega}_t + \frac{1}{n-1} ((\Delta u) \omega - \sqrt{-1} \partial \bar{\partial} u) > 0$  and  $\Omega := e^{\psi/S} \omega^n$ . Note that

$$(2.11) \quad \begin{aligned} \frac{\partial}{\partial t} u &= \log \frac{\left( \hat{\omega}_t + \frac{1}{n-1} ((\Delta u) \omega - \sqrt{-1} \partial \bar{\partial} u) \right)^n}{\omega^n} - \log e^{\psi/S} \\ &= \log \frac{\det * \left( \hat{\omega}_t + \frac{1}{n-1} ((\Delta u) \omega - \sqrt{-1} \partial \bar{\partial} u) \right)}{\det * \omega} - \frac{1}{S} \psi \\ &= \log \frac{\det (\Psi_t + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2})}{\det \omega^{n-1}} - \frac{1}{S} \psi. \end{aligned}$$

Then if we define

$$(2.12) \quad \omega_t^{n-1} := \Psi_t + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2},$$

equations (2.8) and (2.11) show that

$$(2.13) \quad \begin{aligned} \frac{\partial}{\partial t} \omega_t^{n-1} &= \chi \wedge \omega^{n-2} + \sqrt{-1} \partial \bar{\partial} \frac{\partial}{\partial t} u \wedge \omega^{n-2} \\ &= -(n-1) \text{Ric}^C(\omega_t) \wedge \omega^{n-2}. \end{aligned}$$

Conversely, suppose that  $\omega_t^{n-1}$  as defined in (2.12) satisfies (1.1), then

$$\begin{aligned} \sqrt{-1} \partial \bar{\partial} \left( \frac{\partial}{\partial t} u \right) \wedge \omega^{n-2} &= \frac{\partial}{\partial t} (\omega_t^{n-1} - \Psi_t) \\ &= \left( \sqrt{-1} \partial \bar{\partial} \log \frac{\det \omega_t^{n-1}}{\omega^{n-1}} - \frac{1}{S} \sqrt{-1} \partial \bar{\partial} \psi \right) \wedge \omega^{n-2}. \end{aligned}$$

Using the equalities in (2.11), we see that  $\omega_t^{n-1}$  satisfies (1.1) if and only if  $u$  satisfies (1.5).

We define the Hermitian metric  $\tilde{\omega}$  by

$$(2.14) \quad \tilde{\omega}_t := \hat{\omega}_t + \frac{1}{n-1} ((\Delta u) \omega - \sqrt{-1} \partial \bar{\partial} u).$$

To simplify notation we drop the  $t$  subscripts on the metrics and use  $\tilde{\omega}$  and  $\hat{\omega}$  to denote  $\tilde{\omega}_t$  and  $\hat{\omega}_t$ . However,  $\omega$  will still denote the fixed Hermitian metric  $\omega$  and we will not refer to the family of metrics  $\omega_t$  solving (1.1) for the remainder of this paper.

## 3. PRELIMINARY ESTIMATES

We prove uniform bounds for  $u$ ,  $\dot{u}$ , and the volume form  $\tilde{\omega}^n$ . The estimate for  $u$  is actually simpler than in the elliptic case [25, 26] since we can apply the parabolic maximum principle to (1.5).

**Lemma 3.1.** *Suppose  $u$  satisfies (1.5) on  $M \times [0, S)$ . Then there exists a uniform  $C > 0$  such that*

- (1)  $|u| \leq C$
- (2)  $|\dot{u}| \leq C$
- (3)  $\frac{1}{C}\Omega \leq \tilde{\omega}^n \leq C\Omega$

on  $M \times [0, S)$ .

To prove this, we need a maximum principle that will work in this context.

**Lemma 3.2.** *Let  $v$  be a smooth real-valued function on a compact complex manifold  $M$  with Hermitian metric  $\omega$ . Then at a point  $x_0$  where  $v$  achieves a maximum,*

$$(\Delta v)\omega - \sqrt{-1}\partial\bar{\partial}v \leq 0.$$

*Proof.* Choose coordinates at  $x_0$  so that  $g_{i\bar{j}} = \delta_{i\bar{j}}$  and  $v_{i\bar{j}} := \partial_i \partial_{\bar{j}} v = \lambda_i \delta_{i\bar{j}}$ . Since  $x_0$  is where  $v$  attains a maximum  $\lambda_i \leq 0$  for all  $i = 1, \dots, n$ . Then at  $x_0$ ,

$$(\Delta v)g_{i\bar{j}} = \left( \sum_{i=1}^n \lambda_i \right) \delta_{i\bar{j}} \leq \lambda_i \delta_{i\bar{j}} = v_{i\bar{j}}.$$

□

We will also make use of the tensor

$$\Theta^{i\bar{j}} = \frac{1}{n-1} \left( (\text{tr}_{\tilde{g}} g) g^{i\bar{j}} - \tilde{g}^{i\bar{j}} \right) > 0$$

and the operator  $L$  acting on smooth functions  $v$  on  $M$  defined by

$$Lv = \Theta^{i\bar{j}} \partial_i \partial_{\bar{j}} v.$$

Taking trace of (2.14), we have the useful relation

$$(3.15) \quad n = \text{tr}_{\tilde{\omega}} \hat{\omega} + Lu.$$

Using this, we can prove Lemma 3.1 via maximum principle similar to the analogous estimates for the Kähler-Ricci flow (see [21] for example).

*Proof.* For (1), define a quantity  $Q = u - At$  where  $A$  is a constant to be determined later and fix  $0 < t' < S$ . Then suppose that a maximum of  $Q$  on  $M \times [0, t']$  occurs at a point  $(x_0, t_0)$  with  $t_0 > 0$ . Applying the previous

lemma and the usual maximum principle at  $(x_0, t_0)$ ,

$$\begin{aligned}
0 &\leq \frac{\partial}{\partial t} Q \\
&= \log \frac{\left( \hat{\omega} + \frac{1}{n-1} ((\Delta u)\omega - \sqrt{-1}\partial\bar{\partial}u) \right)^n}{\Omega} - A \\
&\leq \log \frac{\hat{\omega}^n}{\Omega} - A \\
&\leq C - A
\end{aligned}$$

where on the last line we used (2.10). Choosing  $A = C + 1$ , we get a contradiction. Since  $t'$  is arbitrary, we conclude that  $Q$  achieves its maximum at  $t_0 = 0$  and so we have a uniform upper bound for  $u$ . The lower bound follows similarly.

For (2), we compute the evolution equation for  $\dot{u}$ . Using (1.5),

$$(3.16) \quad \frac{\partial}{\partial t} \dot{u} = \text{tr}_{\tilde{\omega}} \left( \frac{\partial}{\partial t} \tilde{\omega} \right) = \frac{1}{n-1} \text{tr}_{\tilde{\omega}} ((\text{tr}_{\omega} \chi)\omega - \chi + (\Delta \dot{u})\omega - \sqrt{-1}\partial\bar{\partial}\dot{u})$$

Then we have

$$\begin{aligned}
(3.17) \quad L\dot{u} &= \frac{1}{n-1} \left( (\text{tr}_{\tilde{g}} g) g^{i\bar{j}} - \tilde{g}^{i\bar{j}} \right) \partial_i \partial_{\bar{j}} \dot{u} \\
&= \frac{1}{n-1} ((\Delta \dot{u})(\text{tr}_{\tilde{\omega}} \omega) - \text{tr}_{\tilde{\omega}} \sqrt{-1}\partial\bar{\partial}\dot{u}).
\end{aligned}$$

Now consider the quantity  $Q = (n-1)\dot{u} - Au$  where  $A$  is a constant to be determined. Combining (3.15), (3.16), and (3.17),

$$\left( \frac{\partial}{\partial t} - L \right) Q = (\text{tr}_{\tilde{\omega}} \omega)(\text{tr}_{\omega} \chi) - \text{tr}_{\tilde{\omega}} \chi - A\dot{u} + An - A\text{tr}_{\tilde{\omega}} \hat{\omega}.$$

Using 2.10, we can choose  $A$  large enough so that

$$A\hat{\omega} \geq (\text{tr}_{\omega} \chi)\omega - \chi$$

which gives

$$\left( \frac{\partial}{\partial t} - L \right) Q \leq -A\dot{u} + An.$$

Hence at a point  $(x_0, t_0)$  at which  $Q$  achieves a maximum,  $\dot{u}(x_0, t_0) \leq n$ . Then since  $Q$  is bounded above by its value at  $(x_0, t_0)$ ,

$$\dot{u} \leq \frac{1}{n-1} \left( A \sup_{M \times [0, S]} u + n(n-1) - Au(x_0, t_0) \right) \leq C$$

where for the last inequality we used the above uniform bound for  $u$ .

To prove the lower bound, consider the quantity

$$Q = (n-1)(S - t + \varepsilon)\dot{u} + u + nt$$



where  $\epsilon > 0$  is a constant to be determined. Again applying (3.15), (3.16), and (3.17),

$$\begin{aligned} \left( \frac{\partial}{\partial t} - L \right) Q &= -\dot{u} + (S - t + \epsilon)((\text{tr}_{\tilde{\omega}}\omega)(\text{tr}_{\omega}\chi) - \text{tr}_{\tilde{\omega}}\chi) + \dot{u} - n + \text{tr}_{\tilde{\omega}}\hat{\omega} + n \\ &= \text{tr}_{\tilde{\omega}}(\hat{\omega}_S + \epsilon((\text{tr}_{\omega}\chi)\omega - \chi)) \\ &> 0 \end{aligned}$$

provided we choose  $\epsilon > 0$  small enough. If  $Q$  achieves a minimum at a point  $(x_0, t_0)$  with  $t_0 > 0$ , we have a contradiction. Hence  $Q$  must be bounded from below by its infimum over  $M$  at time  $t = 0$ . When combined with the uniform bound for  $u$ , this gives the lower bound for  $\dot{u}$ .

To finish the lemma, (3) follows immediately from (2) since we have

$$\dot{u} = \log \frac{\tilde{\omega}^n}{\Omega}.$$

□

#### 4. SECOND ORDER ESTIMATE

We obtain a second order estimate for  $u$  in terms of  $\text{tr}_{\omega}\tilde{\omega}$ . This estimate is the parabolic version of the estimates from Hou-Ma-Wu [11] and Tosatti-Weinkove [25, 26] and the proof follows a similar method.

**Lemma 4.1.** *There exists a uniform  $C > 0$  such that*

$$(4.18) \quad \text{tr}_{\omega}\tilde{\omega} \leq C \left( \sup_{M \times [0, S]} |\nabla u|_g^2 + 1 \right)$$

on  $M \times [0, S]$ .

*Proof.* As in [26] we consider the tensor

$$(4.19) \quad \eta_{i\bar{j}} = u_{i\bar{j}} + (\text{tr}_g \hat{g})g_{i\bar{j}} - (n-1)\hat{g}_{i\bar{j}} = (\text{tr}_g \tilde{g})g_{i\bar{j}} - (n-1)\tilde{g}_{i\bar{j}}.$$

Fix a  $t'$  such that  $0 < t' < S$ . Define the quantity

$$H(x, \xi, t) = \log(\eta_{i\bar{j}}\xi^i\bar{\xi}^j) + c \log \left( g^{p\bar{q}}\eta_{i\bar{q}}\eta_{p\bar{j}}\bar{\xi}^i\bar{\xi}^j \right) + \varphi(|\nabla u|_g^2) + \nu(u)$$

for  $x \in M$ ,  $\xi \in T_x^{1,0}M$  a  $g$ -unit vector,  $t \in [0, t']$ , and  $c > 0$  a small constant to be determined. The above functions are

$$\begin{aligned} \varphi(s) &= -\frac{1}{2} \log \left( 1 - \frac{s}{2K} \right), \quad 0 \leq s \leq K-1 \\ \nu(s) &= -A \log \left( 1 + \frac{s}{2L} \right), \quad -L+1 \leq s \leq L-1, \end{aligned}$$

where

$$K = \sup_{M \times [0, t']} |\nabla u|_g^2 + 1, L = \sup_{M \times [0, S]} |u| + 1, A = 3L(C_1 + 1)$$

with  $C_1$  a uniform constant to be determined during the proof. Note that  $L$  is uniformly bounded by Lemma 3.1. This setup is similar to [11, 25, 26],

the difference being that we have a time dependence. Evaluating at  $|\nabla u|_2^2$ , we have the bounds

$$(4.20) \quad 0 \leq \varphi \leq C, \quad 0 < \frac{1}{4K} \leq \varphi' \leq \frac{1}{2K}, \quad \varphi'' = 2(\varphi')^2 > 0$$

and evaluating at  $u$ ,

$$(4.21) \quad |\nu| \leq C, \quad C_1 + 1 = \frac{A}{3L} \leq -\nu' \leq \frac{A}{L}, \quad \frac{2\varepsilon}{1-\varepsilon}(\nu')^2 \leq \nu'', \quad \text{for all } \varepsilon \leq \frac{1}{2A+1}$$

on  $M \times [0, t']$  for uniform  $C > 0$ .

Similar to [11], we define the set

$$W = \left\{ (x, \xi, t) \mid \eta(x, t)_{i\bar{j}} \xi^i \bar{\xi}^j \geq 0, \xi \in T_x^{1,0} M \text{ a } g\text{-unit vector}, t \in [0, t'] \right\}.$$

Then  $W$  is compact,  $H = -\infty$  on the boundary of a cross section  $W_t$  for fixed time  $t$ , and  $H$  is upper semi-continuous on  $W_t$ . Thus if  $H$  has a maximum at a point  $(x_0, \xi_0, t_0)$  in  $W$ ,  $(x_0, \xi_0)$  is in the interior of  $W_{t_0}$ . We assume without loss of generality that  $t_0 > 0$ .

Choose holomorphic coordinates  $(z^1, \dots, z^n)$  centered at  $x_0$  such that at  $(x_0, t_0)$

$$g_{i\bar{j}} = \delta_{i\bar{j}}, \quad \eta_{i\bar{j}} = \delta_{i\bar{j}} \eta_{i\bar{i}}, \quad \eta_{1\bar{1}} \geq \eta_{2\bar{2}} \geq \dots \geq \eta_{n\bar{n}}.$$

From the definition of  $\eta_{i\bar{j}}$

$$\tilde{g}_{i\bar{j}} = \frac{1}{n-1} \left( -\eta_{i\bar{j}} + (\text{tr}_g \tilde{g}) g_{i\bar{j}} \right)$$

so that  $\tilde{g}_{i\bar{j}}$  is also diagonal at  $(x_0, t_0)$  and we may define  $\lambda_i$  by

$$\tilde{g}_{i\bar{j}} = \lambda_i \delta_{i\bar{j}}.$$

at  $(x_0, t_0)$ . Using (4.19),

$$(4.22) \quad \eta_{i\bar{i}} = \sum_{j=1}^n \lambda_j - (n-1)\lambda_i$$

which gives

$$0 < \lambda_1 \leq \dots \leq \lambda_n$$

and

$$(4.23) \quad \frac{1}{n} \text{tr}_\omega \tilde{\omega} \leq \lambda_n \leq \eta_{1\bar{1}} \leq (n-1)\lambda_n \leq (n-1) \text{tr}_\omega \tilde{\omega}.$$

Following [26], choosing  $c < 1/(n-3)$  when  $n > 3$  or  $c$  any positive real number when  $n = 3$ , the quantity

$$\log(\eta_{i\bar{j}} \xi^i \bar{\xi}^j) + c \log \left( g^{p\bar{q}} \eta_{p\bar{q}} \eta_{p\bar{j}} \xi^i \bar{\xi}^j \right)$$

is maximized at  $(x_0, t_0)$  by  $\xi_0 = \partial/\partial z^1$  since  $\eta_{1\bar{1}}$  is the largest eigenvalue of  $\eta_{i\bar{j}}$ . We extend  $\xi_0$  over our coordinate patch to the unit vector field

$$\xi_0 = g_{1\bar{1}}^{-1/2} \frac{\partial}{\partial z^1}.$$

Now we consider the quantity

$$(4.24) \quad Q(x, t) = H(x, \xi_0, t) = \log \left( g_{1\bar{1}}^{-1} \eta_{1\bar{1}} \right) + \varphi \left( |\nabla u|_g^2 \right) + \nu(u)$$

defined in a neighborhood of  $(x_0, t_0)$  chosen small enough so that  $Q$  attains its maximum at  $(x_0, t_0)$ . The proof of the estimate follows from applying the maximum principle to this quantity to obtain the bound

$$(4.25) \quad \eta_{1\bar{1}}(x_0, t_0) \leq CK = C \left( \sup_{M \times [0, t']} |\nabla u|_g^2 + 1 \right).$$

which will complete the proof: at any point  $(x, t) \in M \times [0, t']$  using (4.23),

$$(4.26) \quad \begin{aligned} \mathrm{tr}_\omega \tilde{\omega}(x, t) &\leq n \eta_{1\bar{1}}(x, t) \\ &\leq n \sup_W \left( (\eta_{i\bar{j}} \xi^i \bar{\xi}^j)^{1/(1+2c)} \left( g^{p\bar{q}} \eta_{i\bar{q}} \eta_{p\bar{j}} \xi^i \bar{\xi}^j \right)^{c/(1+2c)} \right) \\ &\leq C e^{Q(x_0, t_0)} \\ &\leq C \left( \sup_{M \times [0, t']} |\nabla u|_g^2 + 1 \right). \end{aligned}$$

Since  $C > 0$  is uniform we get the desired estimate (4.18).

We begin the proof of the estimate (4.25). First, we collect some useful facts. At the point  $(x_0, t_0)$ ,

$$(4.27) \quad \sum_i \Theta^{i\bar{i}} = \mathrm{tr}_{\tilde{g}} g$$

and we may assume that at this point

$$(4.28) \quad |u_{i\bar{j}}| \leq 2|\eta_{1\bar{1}}|$$

since our goal is to prove a uniform bound for  $\eta_{1\bar{1}}(x_0, t_0)$ . As in [26] we have at  $(x_0, t_0)$

$$(4.29) \quad \begin{aligned} L(Q) &\geq (1+2c) \sum_i \frac{\Theta^{i\bar{i}} \eta_{1\bar{1}} \eta_{i\bar{i}}}{\eta_{1\bar{1}}} + \frac{c}{2} \sum_i \sum_{p \neq 1} \frac{\Theta^{i\bar{i}} |\eta_{p\bar{1}}|^2}{(\eta_{1\bar{1}})^2} + \frac{c}{2} \sum_i \sum_{p \neq 1} \frac{\Theta^{i\bar{i}} |\eta_{1\bar{p}}|^2}{(\eta_{1\bar{1}})^2} \\ &\quad - (1+2c) \sum_i \frac{\Theta^{i\bar{i}} |\eta_{1\bar{1}} \eta_{i\bar{i}}|^2}{(\eta_{1\bar{1}})^2} + \nu' \sum_i \Theta^{i\bar{i}} u_{i\bar{i}} + \nu'' \sum_i \Theta^{i\bar{i}} |u_i|^2 \\ &\quad + \varphi'' \sum_i \Theta^{i\bar{i}} \left| \sum_p u_p u_{\bar{p}i} + \sum_p u_{p\bar{i}} u_{\bar{p}} \right|^2 + \varphi' \sum_{i,p} \Theta^{i\bar{i}} \left( |u_{p\bar{i}}|^2 + |u_{p\bar{i}}|^2 \right) \\ &\quad + \varphi' \sum_{i,p} \Theta^{i\bar{i}} \left( u_{p\bar{i}} u_{\bar{p}} + u_{\bar{p}i} u_p \right) - C \mathrm{tr}_{\tilde{g}} g \end{aligned}$$

for a uniform  $C > 0$  where the subscripts denote covariant derivatives with respect to the fixed Hermitian metric  $g$ .

Computing the time evolution of  $Q$  at  $(x_0, t_0)$ ,

$$(4.30) \quad \frac{\partial}{\partial t} Q = (1 + 2c) \frac{\dot{\eta}_{1\bar{1}}}{\eta_{1\bar{1}}} + \varphi' \left( \sum_p \dot{u}_p u_{\bar{p}} + \sum_p \dot{u}_{\bar{p}} u_p \right) + \nu' \dot{u}.$$

Using the definition of  $\eta_{i\bar{j}}$  (4.19),

$$\begin{aligned} \dot{\eta}_{i\bar{j}} &= \dot{u}_{i\bar{j}} + \left( \text{tr}_g \frac{\partial}{\partial t} \hat{g} \right) g_{i\bar{j}} - (n-1) \frac{\partial}{\partial t} \hat{g}_{i\bar{j}} \\ &= \dot{u}_{i\bar{j}} + (\text{tr}_g \chi) g_{i\bar{j}} - \left( (\text{tr}_g \chi) g_{i\bar{j}} - \chi_{i\bar{j}} \right). \end{aligned}$$

Evaluating at  $(x_0, t_0)$ ,

$$(4.31) \quad \dot{\eta}_{1\bar{1}} = \dot{u}_{1\bar{1}} + \chi_{1\bar{1}}.$$

Covariantly differentiating the flow (1.5) with respect to  $g$ ,

$$(4.32) \quad \dot{u}_l = \tilde{g}^{i\bar{j}} \nabla_l \tilde{g}_{i\bar{j}} - \frac{1}{S} \psi_l$$

and

$$(4.33) \quad \dot{u}_{l\bar{m}} = \tilde{g}^{i\bar{j}} \nabla_{\bar{m}} \nabla_l \tilde{g}_{i\bar{j}} - \tilde{g}^{i\bar{q}} \tilde{g}^{p\bar{j}} \nabla_{\bar{m}} \tilde{g}_{p\bar{q}} \nabla_l \tilde{g}_{i\bar{j}} - \frac{1}{S} \psi_{l\bar{m}}.$$

Using the definition of  $\tilde{g}$  (2.14),

$$\dot{u}_l = \Theta^{i\bar{j}} u_{i\bar{j}l} + \tilde{g}^{i\bar{j}} \nabla_l \hat{g}_{i\bar{j}} - \frac{1}{S} \psi_l$$

and letting  $\hat{h}_{i\bar{j}} = (n-1) \hat{g}_{i\bar{j}}$ ,

$$\begin{aligned} \dot{u}_{l\bar{m}} &= \Theta^{i\bar{j}} u_{i\bar{j}l\bar{m}} + \tilde{g}^{i\bar{j}} \nabla_{\bar{m}} \nabla_l \hat{g}_{i\bar{j}} - \frac{1}{S} \psi_{l\bar{m}} \\ &\quad - \frac{\tilde{g}^{i\bar{q}} \tilde{g}^{p\bar{j}} \left( g_{p\bar{q}} g^{r\bar{s}} u_{r\bar{s}\bar{m}} - u_{p\bar{q}\bar{m}} + \nabla_{\bar{m}} \hat{h}_{p\bar{q}} \right) \left( g_{i\bar{j}} g^{r\bar{s}} u_{r\bar{s}l} - u_{i\bar{j}l} + \nabla_l \hat{h}_{i\bar{j}} \right)}{(n-1)^2}. \end{aligned}$$

At  $(x_0, t_0)$ , these become

$$(4.34) \quad \dot{u}_p = \sum_i \Theta^{i\bar{i}} u_{i\bar{i}p} + \sum_i \tilde{g}^{i\bar{i}} \hat{g}_{i\bar{i}p} - \frac{1}{S} \psi_p$$

and

$$(4.35) \quad \dot{u}_{1\bar{1}} = \sum_i \Theta^{i\bar{i}} u_{i\bar{i}1\bar{1}} + \sum_i \tilde{g}^{i\bar{i}} \hat{g}_{i\bar{i}1\bar{1}} - \frac{1}{S} \psi_{1\bar{1}} - H$$

where

$$(4.36) \quad H = \frac{\sum_{i,j} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} \left( g_{j\bar{i}} \sum_a u_{a\bar{a}i} - u_{j\bar{i}1} + \hat{h}_{j\bar{i}1} \right) \left( g_{i\bar{j}} \sum_b u_{b\bar{b}1} - u_{i\bar{j}1} + \hat{h}_{i\bar{j}1} \right)}{(n-1)^2}.$$

Applying the commutation rule (2.6), (4.35) becomes

$$\begin{aligned}
 (4.37) \quad \dot{u}_{1\bar{1}} = & -H + \sum_i \Theta^{i\bar{i}} u_{1\bar{1}i\bar{i}} + \sum_i \tilde{g}^{i\bar{i}} \hat{g}_{i\bar{i}1\bar{1}} - \frac{1}{S} \psi_{1\bar{1}} \\
 & + \sum_i \Theta^{i\bar{i}} \left( u_{p\bar{i}} R_{1\bar{1}i}{}^p - u_{p\bar{1}} R_{i\bar{i}1}{}^p \right) \\
 & - \sum_i \Theta^{i\bar{i}} \left( T_{1i}^p u_{p\bar{1}i} + \overline{T_{1i}^p} u_{1\bar{p}i} + T_{i1}^p \overline{T_{1i}^q} u_{p\bar{q}} \right).
 \end{aligned}$$

Combining (4.30), (4.31), (4.37), and the fact that

$$u_{1\bar{1}i\bar{i}} = \eta_{1\bar{1}i\bar{i}} + \hat{h}_{1\bar{1}i\bar{i}} - (\text{tr}_g \hat{g})_{i\bar{i}}$$

we have the evolution equation

$$\begin{aligned}
 (4.38) \quad \frac{\partial}{\partial t} Q = & -(1+2c) \frac{H}{\eta_{1\bar{1}}} + (1+2c) \sum_i \frac{\Theta^{i\bar{i}} \eta_{1\bar{1}i\bar{i}}}{\eta_{1\bar{1}}} \\
 & + \frac{1+2c}{\eta_{1\bar{1}}} \left( \chi_{1\bar{1}} - \frac{1}{S} \psi_{1\bar{1}} + \sum_i \Theta^{i\bar{i}} \left( u_{p\bar{i}} R_{1\bar{1}i}{}^p - u_{p\bar{1}} R_{i\bar{i}1}{}^p \right) \right. \\
 & + \sum_i \tilde{g}^{i\bar{i}} \hat{g}_{i\bar{i}1\bar{1}} + \sum_i \Theta^{i\bar{i}} \left( \hat{h}_{1\bar{1}i\bar{i}} - (\text{tr}_g \hat{g})_{i\bar{i}} \right) \Big) \\
 & - \frac{2(1+2c)}{\eta_{1\bar{1}}} \sum_{i,p} \Theta^{i\bar{i}} \text{Re} \left( \overline{T_{1i}^p} u_{p\bar{1}i} \right) - \frac{1+2c}{\eta_{1\bar{1}}} \sum_{i,p} \Theta^{i\bar{i}} T_{i1}^p \overline{T_{1i}^q} u_{p\bar{q}} \\
 & + \varphi' \left( \sum_p \dot{u}_p u_{\bar{p}} + \sum_p \dot{u}_{\bar{p}} u_p \right) + \nu' \dot{u}.
 \end{aligned}$$

Subtracting (4.38) and (4.29) we obtain the evolution equation bound at  $(x_0, t_0)$ ,

(4.39)

$$\begin{aligned}
0 &\leq \left( \frac{\partial}{\partial t} - L \right) Q \\
&\leq -(1+2c) \frac{H}{\eta_{1\bar{1}}} \\
&\quad - \frac{c}{2} \sum_i \sum_{p \neq 1} \frac{\Theta^{i\bar{i}} |\eta_{p\bar{1}i}|^2}{(\eta_{1\bar{1}})^2} - \frac{c}{2} \sum_i \sum_{p \neq 1} \frac{\Theta^{i\bar{i}} |\eta_{1\bar{p}i}|^2}{(\eta_{1\bar{1}})^2} + (1+2c) \sum_i \frac{\Theta^{i\bar{i}} |\eta_{1\bar{1}i}|^2}{(\eta_{1\bar{1}})^2} \\
&\quad + C \operatorname{tr}_{\hat{g}} g + \frac{1+2c}{\eta_{1\bar{1}}} \left( \chi_{1\bar{1}} - \frac{1}{S} \psi_{1\bar{1}} + \sum_i \Theta^{i\bar{i}} \left( u_{p\bar{i}} R_{1\bar{1}i}{}^p - u_{p\bar{1}} R_{i\bar{1}}{}^p \right) \right) \\
&\quad + \sum_i \hat{g}^{i\bar{i}} \hat{g}_{i\bar{1}\bar{1}} + \sum_i \Theta^{i\bar{i}} \left( \hat{h}_{1\bar{1}i\bar{i}} - (\operatorname{tr}_g \hat{g})_{i\bar{i}} \right) \\
&\quad - \frac{2(1+2c)}{\eta_{1\bar{1}}} \sum_{i,p} \Theta^{i\bar{i}} \operatorname{Re} \left( \overline{T_{1i}^p} u_{p\bar{1}i} \right) - \frac{1+2c}{\eta_{1\bar{1}}} \sum_{i,p} \Theta^{i\bar{i}} T_{i\bar{1}}^p \overline{T_{1i}^q} u_{p\bar{q}} \\
&\quad + \nu' \left( \frac{\partial}{\partial t} - L \right) u \\
&\quad - \nu'' \sum_i \Theta^{i\bar{i}} |u_i|^2 - \varphi'' \sum_i \Theta^{i\bar{i}} \left| \sum_p u_p u_{\bar{p}i} + \sum_p u_{pi} u_{\bar{p}} \right|^2 \\
&\quad - \varphi' \sum_{i,p} \Theta^{i\bar{i}} \left( |u_{p\bar{i}}|^2 + |u_{pi}|^2 \right) \\
&\quad + \varphi' \sum_p \left( \left( \dot{u}_p - \sum_i \Theta^{i\bar{i}} u_{pi\bar{i}} \right) u_{\bar{p}} + u_{\bar{p}} \left( \dot{u}_{\bar{p}} - \sum_i \Theta^{i\bar{i}} u_{\bar{p}i\bar{i}} \right) \right) \\
&= (1) + (2) + (3) + (4) + (5) + (6) + (7) + (8) + (9)
\end{aligned}$$

where (1) through (9) correspond to the lines in the last inequality. We now bound each of the lines of (4.39) from above.

Lines (3) and (4): Using (4.27) and (4.28) we have the upper bound

$$(3) + (4) \leq C \operatorname{tr}_{\hat{g}} g + C.$$

Line (5): As in [26], using the second term from line (2) we can bound line (5). Covariantly differentiating (4.19),

$$u_{1\bar{p}i} = \eta_{1\bar{p}i} - (\operatorname{tr}_g \hat{g})_i g_{1\bar{p}} + \hat{h}_{1\bar{p}i}$$

and so

$$(4.40) \quad -\frac{2(1+2c)}{\eta_{1\bar{1}}} \sum_{i,p} \Theta^{i\bar{i}} \operatorname{Re} \left( \overline{T_{1i}^p} u_{p\bar{1}i} \right) \leq -\frac{2(1+2c)}{\eta_{1\bar{1}}} \sum_{i,p} \Theta^{i\bar{i}} \operatorname{Re} \left( \overline{T_{1i}^p} \eta_{p\bar{1}i} \right) + C \operatorname{tr}_{\hat{g}} g.$$

Since  $T_{11}^1 = 0$ , the term from the sum with  $p = 1$  is

$$(4.41) \quad -\frac{2(1+2c)}{\eta_{1\bar{1}}} \sum_i \Theta^{i\bar{i}} \operatorname{Re} \left( \overline{T_{1i}^1} \eta_{1\bar{1}i} \right) = -\frac{2(1+2c)}{\eta_{1\bar{1}}} \sum_{i \neq 1} \Theta^{i\bar{i}} \operatorname{Re} \left( \overline{T_{1i}^1} \eta_{1\bar{1}i} \right).$$

The remaining summands can be bounded by

$$(4.42) \quad -\frac{2(1+2c)}{\eta_{1\bar{1}}} \sum_i \sum_{p \neq 1} \Theta^{i\bar{i}} \operatorname{Re} \left( \overline{T_{1i}^p} \eta_{p\bar{1}i} \right) \leq \frac{c}{4} \sum_i \sum_{p \neq 1} \Theta^{i\bar{i}} \frac{|\eta_{1\bar{p}i}|^2}{(\eta_{1\bar{1}})^2} + C \operatorname{tr}_{\hat{g}} g.$$

Putting together (4.40), (4.41), (4.42) and controlling the second term in (6) using (4.28) we have the bound

$$(6) \leq -\frac{2(1+2c)}{\eta_{1\bar{1}}} \sum_{i \neq 1} \Theta^{i\bar{i}} \operatorname{Re} \left( \overline{T_{1i}^1} \eta_{1\bar{1}i} \right) + \frac{c}{4} \sum_i \sum_{p \neq 1} \Theta^{i\bar{i}} \frac{|\eta_{1\bar{p}i}|^2}{(\eta_{1\bar{1}})^2} + C \operatorname{tr}_{\hat{g}} g.$$

Line (6): Applying (3.15), the uniform bound for  $\dot{u}$ , and (4.21),

$$\begin{aligned} (6) &= \nu' \dot{u} - n\nu' + \nu' \operatorname{tr}_{\hat{g}} \hat{g} \\ &\leq 3C(C_1 + 1) + 3(C_1 + 1)n - (C_1 + 1) \operatorname{tr}_{\hat{g}} \hat{g} \\ &\leq C - (C_1 + 1) \operatorname{tr}_{\hat{g}} \hat{g} \end{aligned}$$

remembering that  $C_1 > 0$  is to be determined.

Lines (8) and (9): For line (9), commuting covariant derivatives and recalling (4.34)

$$\begin{aligned} (9) &= \varphi' \sum_p \left( \left( \dot{u}_p - \sum_i \Theta^{i\bar{i}} u_{i\bar{i}p} \right) u_{\bar{p}} + u_{\bar{p}} \left( \dot{u}_{\bar{p}} - \sum_i \Theta^{i\bar{i}} u_{i\bar{i}\bar{p}} \right) \right) \\ &\quad - \varphi' \sum_{i,p} \Theta^{i\bar{i}} u_q u_{\bar{p}} R_{i\bar{i}p}^q + 2 \operatorname{Re} \varphi' \sum_{i,p,q} \Theta^{i\bar{i}} u_{\bar{p}} u_{q\bar{i}} T_{ip}^q \\ &= \varphi' \sum_{i,p} \tilde{g}^{i\bar{i}} \left( \hat{g}_{i\bar{i}p} u_{\bar{p}} + \hat{g}_{i\bar{i}\bar{p}} u_p \right) - \varphi' \sum_p \left( \frac{\psi_p}{S} u_{\bar{p}} + \frac{\psi_{\bar{p}}}{S} u_p \right) \\ &\quad - \varphi' \sum_{i,p} \Theta^{i\bar{i}} u_q u_{\bar{p}} R_{i\bar{i}p}^q + 2 \operatorname{Re} \varphi' \sum_{i,p,q} \Theta^{i\bar{i}} u_{\bar{p}} u_{q\bar{i}} T_{ip}^q. \end{aligned}$$

Thankfully,  $\varphi'$  can be used to control the single derivatives of  $u$  via (4.20). Combining this and (4.27),

$$(9) \leq C + C \operatorname{tr}_{\hat{g}} g + \frac{1}{10} \varphi' \sum_{i,p} \Theta^{i\bar{i}} \left( |u_{p\bar{i}}|^2 + |u_{p\bar{i}}|^2 \right).$$

Together with (8) we have the upper bound

$$(8) + (9) \leq C + C \operatorname{tr}_{\hat{g}} g - \frac{9}{10} \varphi' \sum_{i,p} \Theta^{i\bar{i}} \left( |u_{p\bar{i}}|^2 + |u_{pi}|^2 \right).$$

Combining the above estimates for the lines in (4.39), we have

$$\begin{aligned} 0 &\leq -(1+2c) \frac{H}{\eta_{1\bar{1}}} \\ &\quad - \frac{c}{2} \sum_i \sum_{p \neq 1} \frac{\Theta^{i\bar{i}} |\eta_{p\bar{1}i}|^2}{(\eta_{1\bar{1}})^2} - \frac{c}{4} \sum_i \sum_{p \neq 1} \frac{\Theta^{i\bar{i}} |\eta_{1\bar{p}i}|^2}{(\eta_{1\bar{1}})^2} + (1+2c) \sum_i \frac{\Theta^{i\bar{i}} |\eta_{1\bar{1}i}|^2}{(\eta_{1\bar{1}})^2} \\ &\quad - \nu'' \sum_i \Theta^{i\bar{i}} |u_i|^2 - \varphi'' \sum_i \Theta^{i\bar{i}} \left| \sum_p u_p u_{\bar{p}i} + \sum_p u_{pi} u_{\bar{p}} \right|^2 \\ &\quad + C + C_0 \operatorname{tr}_{\hat{g}} g - \frac{9}{10} \varphi' \sum_{i,p} \Theta^{i\bar{i}} \left( |u_{p\bar{i}}|^2 + |u_{pi}|^2 \right) \\ &\quad - \frac{2(1+2c)}{\eta_{1\bar{1}}} \sum_{i \neq 1} \Theta^{i\bar{i}} \operatorname{Re} \left( \overline{T_{1i}} \eta_{1\bar{1}i} \right) - (C_1 + 1) \operatorname{tr}_{\hat{g}} \hat{g}. \end{aligned}$$

This is the same inequality as part way through the second order estimate in [26]. Since we are fixed at the point  $(x_0, t_0)$ ,  $\hat{g}$  is a fixed Hermitian metric. This lets us choose  $C_1 > 0$  uniform and large such that

$$(C_0 + 2) \operatorname{tr}_{\hat{g}} g \leq (C_1 + 1) \operatorname{tr}_{\hat{g}} \hat{g}.$$

The remainder of the estimate goes through exactly as in [26] and we will not reproduce it here. This gives the bound

$$\eta_{1\bar{1}}(x_0, t_0) \leq CK$$

for uniform  $C > 0$  which completes the proof as discussed above.  $\square$

## 5. FIRST ORDER ESTIMATE

Given the form of our second order estimate we require a first order estimate for  $u$ . For the proof we modify the argument of [25] to apply in this parabolic setting.

**Lemma 5.1.** *There exists a uniform  $C > 0$  such that*

$$(5.43) \quad \sup_{M \times [0, S)} |\nabla u|_g^2 \leq C.$$

The proof of this lemma requires a bit of machinery which we will recall from [25]. Let  $\beta$  be the Euclidean Kähler form on  $\mathbb{C}^n$  and  $\Delta$  the Laplacian with respect to  $\beta$ . Let  $\Omega \subset \mathbb{C}^n$  be a domain. We say that an upper semi-continuous function

$$u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$$



in  $L^1_{loc}(\Omega)$  is  $(n-1)$ -PSH if

$$P(u) := \frac{1}{n-1} ((\Delta u)\beta - \sqrt{-1}\partial\bar{\partial}u) \geq 0$$

as a  $(1,1)$ -current. A continuous  $(n-1)$ -PSH function  $u$  is maximal if for any relatively compact open set  $\Omega' \Subset \Omega$  and any continuous  $(n-1)$ -PSH function  $v$  on a domain  $\Omega' \Subset \Omega'' \Subset \Omega$  and with  $v \leq u$  on  $\partial\Omega'$ , then  $v \leq u$  on  $\Omega'$ .

We need the following Liouville-type theorem from [25].

**Theorem 5.2.** (*Tosatti-Weinkove*) *If  $u : \mathbb{C}^n \rightarrow \mathbb{R}$  is an  $(n-1)$ -PSH function in  $\mathbb{C}^n$  which is Lipschitz continuous, maximal, and satisfies*

$$\sup_{\mathbb{C}^n} (|u| + |\nabla u|) < \infty$$

*then  $u$  is constant.*

The proof of this result uses an idea of Dinew-Kołodziej [1]. With these definitions and the Liouville-type theorem, we now begin the proof of Lemma 5.1.

*Proof.* Suppose for contradiction that (5.43) does not hold. Then there exists a sequence  $(x_j, t_j) \in M \times [0, S]$  with  $t_j \rightarrow S$  such that

$$\lim_{j \rightarrow \infty} |\nabla u(x_j, t_j)|_g^2 = \infty.$$

Without loss of generality we assume our  $t_j$  are such that

$$\sup_{x \in M} |\nabla u(x, t_j)|_g^2 = \sup_{M \times [0, t_j]} |\nabla u|_g^2.$$

Additionally, we choose our  $x_j$  to be a point at which  $|\nabla u(\cdot, t_j)|_g$  attains its maximum. We define

$$C_j := |\nabla u(x_j, t_j)|_g^2 = \sup_{M \times [0, t_j]} |\nabla u|_g^2$$

which has the property  $C_j \rightarrow \infty$  as  $j \rightarrow \infty$ .

With this setup, we are ready to apply the blow-up argument and the Liouville-type theorem from [25] to obtain a contradiction. After passing to a subsequence, there exists an  $x$  in  $M$  such that  $x_j \rightarrow x$  as  $j \rightarrow \infty$ . Fix holomorphic coordinates  $(z^1, \dots, z^n)$  centered at  $x$  with  $\omega(x) = \beta$  and identifying with the ball  $B_2(0) \subset \mathbb{C}^n$ . Also assume that  $j$  is sufficiently large so that  $x_j \in B_1(0)$ . We define

$$\begin{aligned} u_j(x) &= u(x, t_j) \\ \Phi_j(z) &= C_j^{-1}z + x_j \end{aligned}$$

and

$$\hat{u}_j(z) := (u_j \circ \Phi_j)(z) = u_j(C_j^{-1}z + x_j) \text{ for } z \in B_{C_j}(0).$$

Note that by construction  $\hat{u}_j$  achieves its maximum at  $z = 0$  and

$$(5.44) \quad |\nabla \hat{u}_j|_\beta(0) = C_j^{-1} |\nabla u(x_j)|_g = 1.$$

We also have the uniform bounds

$$\sup_{B_{C_j}(0)} |\hat{u}_j|_\beta \leq C, \quad \sup_{B_{C_j}(0)} |\nabla \hat{u}_j|_\beta \leq 1.$$

Using Lemma 4.1 on  $[0, t_j]$  (see (4.26))

$$\sup_{y \in M} |\sqrt{-1} \partial \bar{\partial} u(y, t_j)|_g \leq C' \left( \sup_{M \times [0, t_j]} |\nabla u|_g^2 + 1 \right) = C' C_j^2 + C'$$

which gives the estimate

$$\sup_{B_{C_j}(0)} |\sqrt{-1} \partial \bar{\partial} \hat{u}_j|_\beta \leq \frac{C}{C_j^2} \sup_{y \in M} |\sqrt{-1} \partial \bar{\partial} u(y, t_j)|_g \leq C''.$$

For every compact  $K \subset \mathbb{C}^n$ , every  $0 < \alpha < 1$ , and every  $p > 1$  there exists uniform  $C > 0$  such that

$$\|\hat{u}_j\|_{C^{1,\alpha}(K)} + \|\hat{u}_j\|_{W^{2,p}(K)} \leq C$$

using the Sobolev embedding theorem. From this we have a function  $u \in W_{loc}^{2,p}(\mathbb{C}^n)$  such that a subsequence  $\hat{u}_j$  converges strongly in  $C_{loc}^{1,\alpha}(\mathbb{C}^n)$  and weakly in  $W_{loc}^{2,p}(\mathbb{C}^n)$  to  $u$ . Thus from the estimates for  $\hat{u}_j$  we have the uniform bounds

$$\sup_{\mathbb{C}^n} (|u| + |\nabla u|) \leq C$$

and from (5.44)  $u$  is nonconstant. Following the remainder of the argument for the elliptic case in [25] shows that  $u$  is maximal and is hence constant by the Liouville-type theorem, a contradiction.  $\square$

## 6. HIGHER ORDER ESTIMATES AND PROOF OF THE MAIN THEOREM

To finish the proof of the main theorem, it sufficed to prove the uniform higher order estimates

$$\|u\|_{C^k(M,g)} \leq C_k$$

for  $k = 0, 1, 2, \dots$ . With these estimates the flow converges smoothly as  $t \rightarrow S$  to a metric  $\omega_S$ . We extend the flow to  $[0, S]$  with  $\omega_t|_{t=S} = \omega_S$  allowing us to begin the flow once more. This contradicts the fact that  $S$  is maximal so we must have  $S = T$  since the flow cannot exist beyond  $T$ . We now prove the higher order estimates.

Summarizing our current estimates for  $u$ , we have

$$\sup_{M \times [0, S]} |u| + \sup_{M \times [0, S]} |\nabla u|_g + \sup_{M \times [0, S]} |\sqrt{-1} \partial \bar{\partial} u|_g + \sup_{M \times [0, S]} |\dot{u}|_g \leq C$$

for a uniform  $C > 0$ . Note that from the volume form bound in Lemma 3.1 and the trace bound in Lemma 4.1 we have that  $\tilde{g}$  is uniformly equivalent to  $g$ :

$$(6.45) \quad \frac{1}{C} g \leq \tilde{g} \leq C g.$$

Using standard parabolic theory, the higher order estimates follow from a uniform parabolic  $C^{2+\alpha}(M, g)$  bound for  $u$  for some  $\alpha > 0$ . This can be done via the parabolic Evans-Krylov method as in [8] with some modification (also see [7, 15]).

Let  $B_R$  be a small ball in  $\mathbb{C}^n$  of radius  $R > 0$  centered at the origin. Let  $\varepsilon > 0$  and fix  $t_0 \in [\varepsilon, T)$ . We work in the parabolic cylinder

$$Q(R, t_0) = \{(x, t) \in B_R \times [0, S) \mid t_0 - R^2 \leq t \leq t_0\}.$$

Let  $\{\gamma_i\}$  be a basis for  $\mathbb{C}^n$ . For the  $C^{2+\alpha}(M, g)$  estimate it suffices to prove the bound

$$\sum_{i=1}^n \text{osc}_{Q(R, t_0)}(u_{\gamma_i \bar{\gamma}_i}) + \text{osc}_{Q(R, t_0)}(\dot{u}) \leq CR^\delta$$

for any  $t_0 \in [\varepsilon, S)$ , for some uniform  $C > 0$ , some  $R > 0$  sufficiently small, and some  $\delta > 0$ .

We first rewrite the flow (1.5) as

$$(6.46) \quad -\frac{\partial}{\partial t}u + \log \det \tilde{g} = \tilde{F}$$

where  $\tilde{F} = \psi/S + \log \Omega$ . Let  $\gamma$  be an arbitrary unit vector in  $\mathbb{C}^n$ . We differentiate the flow covariantly and commute derivatives as in (4.35) and (4.37) to obtain

$$-\frac{\partial}{\partial t}u_{\gamma \bar{\gamma}} + \Theta^{i\bar{j}}u_{i\bar{j}\gamma \bar{\gamma}} \geq G + \frac{H}{\eta_{1\bar{1}}} - C \sum_{p,q} |u_{p\bar{q}\gamma}|$$

where  $G$  is bounded function (using our existing uniform estimates) and

$$H = \frac{\tilde{g}^{i\bar{q}}\tilde{g}^{p\bar{j}} \left( g_{p\bar{q}}g^{r\bar{s}}u_{r\bar{s}\gamma} - u_{p\bar{q}\gamma} + \hat{h}_{p\bar{q}\gamma} \right) \left( g_{i\bar{j}}g^{a\bar{b}}u_{a\bar{b}\gamma} - u_{i\bar{j}\gamma} + \hat{h}_{i\bar{j}\gamma} \right)}{(n-1)^2}$$

as in (4.36). Converting the covariant derivatives to partial derivatives,

$$-\frac{\partial}{\partial t}u_{\gamma \bar{\gamma}} + \Theta^{i\bar{j}}\partial_i\partial_{\bar{j}}u_{\gamma \bar{\gamma}} \geq G + H - C \sum_{p,q} |u_{p\bar{q}\gamma}|$$

for a larger  $C > 0$ . The latter two terms cancel because we have the estimate

$$\begin{aligned} \frac{H}{\eta_{1\bar{1}}} &\geq \frac{1}{C'} \tilde{g}^{i\bar{q}}\tilde{g}^{p\bar{j}} \left( g_{p\bar{q}}g^{r\bar{s}}u_{r\bar{s}\gamma} - u_{p\bar{q}\gamma} \right) \left( g_{i\bar{j}}g^{a\bar{b}}u_{a\bar{b}\gamma} - u_{i\bar{j}\gamma} \right) - C' \\ &\geq \frac{1}{C'} \left( (n-2)|g^{r\bar{s}}u_{r\bar{s}\gamma}|^2 + \tilde{g}^{i\bar{q}}\tilde{g}^{p\bar{j}}u_{i\bar{j}\gamma}u_{p\bar{q}\gamma} \right) - C' \\ &\geq C \sum_{p,q} |u_{p\bar{q}\gamma}| - C' \end{aligned}$$

for a uniform constant  $C' > 0$ , giving the bound

$$(6.47) \quad -\frac{\partial}{\partial t}u_{\gamma \bar{\gamma}} + \Theta^{i\bar{j}}\partial_i\partial_{\bar{j}}u_{\gamma \bar{\gamma}} \geq G.$$

We also have

$$(6.48) \quad -\frac{\partial}{\partial t}\dot{u} + \Theta^{i\bar{j}}\partial_i\partial_{\bar{j}}\dot{u} = \frac{(\mathrm{tr}_{\tilde{g}}g)(\mathrm{tr}_g\chi) - \mathrm{tr}_{\tilde{g}}\chi}{n-1} \leq C$$

using (3.16), (3.17), Lemma 4.1, and Lemma 5.1 for a uniform  $C > 0$ .

As in [25, 26] we define a metric  $g'_{i\bar{j}} = g'_{i\bar{j}}(x_0)$  on  $B_R$ . This fixed metric allows us to contract tensors that would otherwise be at different points in space and time. We will also use the tensor

$$\hat{\Theta}^{i\bar{j}} = \frac{1}{n-1} \left( (\mathrm{tr}_{\tilde{g}}g')g'^{i\bar{j}} - \tilde{g}^{i\bar{j}} \right)$$

and the operator

$$\Delta' = g'^{i\bar{j}}\partial_i\partial_{\bar{j}}.$$

By the mean value inequality, for all  $x$  in  $B_R$ ,

$$(6.49) \quad |g'_{i\bar{j}}(x) - g_{i\bar{j}}(x)| \leq CR.$$

We let  $\Phi$  be an operator on a matrix  $A$  given by  $\Phi(A) = \log \det A$ . Since  $\Phi$  is concave, for all  $(x, t), (y, s) \in B_R \times [0, S]$

$$(6.50) \quad \sum_{i,j} \frac{\partial \Phi}{\partial a_{i\bar{j}}}(\tilde{g}(y, s)) \left( \tilde{g}_{i\bar{j}}(x, t) - \tilde{g}_{i\bar{j}}(y, s) \right) \geq \Phi(\tilde{g}(x, t)) - \Phi(\tilde{g}(y, s)).$$

Using (6.46), equation (6.50) becomes

$$(6.51) \quad \dot{u}(x, t) - \dot{u}(y, s) + \sum_{i,j} \tilde{g}^{i\bar{j}}(y, s) \left( \tilde{g}_{i\bar{j}}(y, s) - \tilde{g}_{i\bar{j}}(x, t) \right) \leq CR$$

after applying the mean value inequality to  $\tilde{F}$ . We need to further bound the last term on the left hand side. Computing from the definition of  $\tilde{g}$

$$(6.52) \quad \begin{aligned} \sum_{i,j} \tilde{g}^{i\bar{j}}(y, s) \left( \tilde{g}_{i\bar{j}}(y, s) - \tilde{g}_{i\bar{j}}(x, t) \right) &= \sum_{i,j} \tilde{g}^{i\bar{j}}(y, s) \left( \hat{g}_{i\bar{j}}(y, s) - \hat{g}_{i\bar{j}}(x, t) \right) \\ &+ \frac{1}{n-1} \sum_{i,j} \tilde{g}^{i\bar{j}}(y, s) \left( ((\Delta u)g_{i\bar{j}} - u_{i\bar{j}})(y, s) - ((\Delta u)g_{i\bar{j}} - u_{i\bar{j}})(x, t) \right). \end{aligned}$$

The mean value inequality in  $Q(R, t_0)$  along with the uniform bounds for  $\tilde{g}$  and  $\hat{g}$  gives

$$(6.53) \quad \left| \sum_{i,j} \tilde{g}^{i\bar{j}}(y, s) \left( \hat{g}_{i\bar{j}}(y, s) - \hat{g}_{i\bar{j}}(x, t) \right) \right| \leq CR.$$

Then with (6.52), (6.53), and the uniform bounds for  $u_{i\bar{j}}$  and  $\tilde{g}_{i\bar{j}}$  equation (6.51) becomes

$$(6.54) \quad \begin{aligned} \frac{1}{n-1} \sum_{i,j} \tilde{g}^{i\bar{j}}(y, s) \left( ((\Delta' u)g'_{i\bar{j}} - u_{i\bar{j}})(y, s) - ((\Delta' u)g'_{i\bar{j}} - u_{i\bar{j}})(x, t) \right) \\ + \dot{u}(x, t) - \dot{u}(y, s) \leq CR. \end{aligned}$$

Here is where we use the fixed metric  $g'$ . Since

$$\sum_{i,j} \hat{\Theta}^{i\bar{j}}(y,s) u_{i\bar{j}}(z,r) = \sum_{i\bar{j}} \tilde{g}^{i\bar{j}}(y,s) \left( (\Delta' u) g'_{i\bar{j}} - u_{i\bar{j}} \right) (z,r),$$

for any  $(z,r) \in B_R \times [0,S)$  we have the estimate

$$(6.55) \quad \dot{u}(x,t) - \dot{u}(y,s) + \sum_{i,j} \hat{\Theta}^{i\bar{j}}(y,s) \left( u_{i\bar{j}}(y,s) - u_{i\bar{j}}(x,t) \right) \leq CR.$$

Following [8] (or [7, 15]) we find finitely many unit vectors  $\gamma_1, \dots, \gamma_n$  in  $\mathbb{C}^n$  and real valued functions  $\beta_\nu$  on  $B_R \times [0,S)$  with

$$0 < C^{-1} < \beta_\nu < C$$

for  $\nu = 1, \dots, N$  such that

$$\hat{\Theta}^{i\bar{j}}(y,s) = \sum_{\nu=1}^N \beta_\nu(y,s) (\gamma_\nu)^i \overline{(\gamma_\nu)^j}.$$

For  $\nu = 1, \dots, N$  define

$$w_\nu = u_{\gamma_\nu \overline{\gamma_\nu}}$$

and for  $\nu = 0$ ,

$$w_0 = -\dot{u}, \text{ and } \beta_0 = 1.$$

From (6.55),

$$(6.56) \quad \sum_{\nu=0}^N \beta_\nu(y,s) (w_\nu(y,s) - w_\nu(x,t)) \leq CR$$

and for all  $\nu = 0, 1, \dots, N$ ,

$$(6.57) \quad -\frac{\partial}{\partial t} w_\nu + \Theta^{i\bar{j}} \partial_i \partial_{\bar{j}} w_\nu \geq G$$

where  $G$  is a uniformly bounded function using (6.47) and (6.48). With the key estimates (6.56) and (6.57) we can complete the  $C^{2+\alpha}(M,g)$  estimate exactly as in [8] for the parabolic complex Monge-Ampère equation. This finishes the proof of the main theorem.

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